Extensions of line bundles

X irred. proj. variety, L and M line bundles on X

An extension of L by M is a short exact sequence

$$0 \to M \to E \to L \to 0 \quad (*)$$

so that E is a rank 2 vector bundle on X.

$0 \to M \to E' \to L \to 0$ is <u>equivalent</u> to $(*)$ if
$\exists$ $E \to E'$ or $E' \to E$ inducing the identity on M and L.

Set of equiv. classes is $\mathrm{Ext}^1(L, M)$. ($0 \in \mathrm{Ext}^1(L,M)$ corr. to split sequence.)

Given $(*)$, twist by L^*:

$0 \to M \otimes L^* \to E \otimes L^* \to \mathcal{O}_X \to 0$. Then taking cohomology, get a map $\delta_{(*)}: H^0(\mathcal{O}_X) \to H^1(M \otimes L^*)$. This determines a morphism

$$\begin{aligned} \mathrm{Ext}^1(L,M) &\longrightarrow H^1(M \otimes L^*) \\ (*) &\longmapsto \delta_{(*)}(1) \end{aligned}$$

<u>Exercise</u>: Show this is an isomorphism.

<u>Notice</u>: Have a natural map

$$H^0(L) \otimes H^0(L^* \otimes M) \to H^0(M)$$

$H^0(L) \otimes H^0(L^* \otimes \_\_)$ is a functor whose 1st derived functor is $H^0(L) \otimes H^1(L^* \otimes \_\_)$, so get a natural map

$$H^0(L) \otimes H^1(L^* \otimes M) \to H^1(M)$$

equivalently, $H^1(L^* \otimes M) \to \mathrm{Hom}(H^0(L), H^1(M))$

(This is the map sending ext class to its connecting homomorphism)

[[Defs of projective normality + normal generation ($\mathrm{Sym}^m H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \twoheadrightarrow H^0(\mathcal{O}_X(m))$)]]

<u>Example</u>: (Noether's Theorem)

C a sm. projective curve of genus $g \geq 2$. If C is not hyperelliptic, then C is <u>projectively normal</u> in its canonical embedding $\phi_{|K_C|} : C \hookrightarrow \mathbb{P}^{g-1}$

i.e. $\alpha_m : \mathrm{Sym}^m(H^0(K_C)) \to H^0(mK_C)$ is surjective for $m \geq 2$.

<u>Pf for m=2 case</u>: Assume α_2 is not surjective. Then

$$H^0(K_C) \otimes H^0(K_C) \to H^0(2K_C) \text{ is not surj, so (Serre duality)}$$

$$\varphi : H^1(-K_C) \to H^0(K)^* \otimes H^1(\mathcal{O}) \text{ has a kernel.}$$

$$H^0(K)^* \otimes H^1(\mathcal{O}) \cong \mathrm{Hom}(H^0(K), H^1(\mathcal{O}))$$

$\Rightarrow \exists$ nonsplit extension $0 \to \mathcal{O} \to E \to \mathcal{O}(K) \to 0$ (1)

whose connecting homomorphism is 0.

Thus $h^0(E) = h^0(\mathcal{O}) + h^0(K) = 1 + g$.

And since $\mathcal{O}$ and $\mathcal{O}(K)$ are globally generated,

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}) \arrow[r] \arrow[d, two heads] & H^0(E) \arrow[r] \arrow[d] & H^0(\mathcal{O}(K)) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & E \arrow[r] & \mathcal{O}(K) \arrow[r] & 0
\end{tikzcd}$$

so E is globally generated.

Take $P \in C$. Then $\underset{\dim\, g+1 \geq 3}{H^0(E)} \to \underset{\dim 2}{H^0(E|_p)}$ has a kernel,

so $\exists$ $0 \neq s \in H^0(E)$ vanishing at P

Let $D \subseteq C$ be the vanishing locus of s.

Then we have SES

$$\begin{tikzcd}
0 \arrow[r] & E(-D) \arrow[r] & E \arrow[r] & E|_D \arrow[r] & 0 \\
& & \mathcal{O} \arrow[u, "s"'] \arrow[ul] & &
\end{tikzcd}$$

(vanishes on D so)

get

$$0 \to \mathcal{O} \to E(-D) \to \mathcal{O}(K-2D) \to 0 \quad \text{twist by } D\ldots$$

$$0 \to \mathcal{O}(D) \to E \to \mathcal{O}(K-D) \to 0 \qquad (2)$$

$$\text{so } 1+g = h^0(E) \leq h^0(\mathcal{O}(D)) + h^0(\mathcal{O}(K-D))$$
$$= 2h^0(\mathcal{O}(D)) + 2g - 2 - \deg(D)$$
$$\Longrightarrow \deg(D) \leq 2\left(h^0(\mathcal{O}(D)) - 1\right) = 2 \dim |D|$$

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<u>Clifford's Theorem</u>: D an effective special $\left(h^0(K-D) > 0 \right)$ divisor on C. Then

$$\deg(D) \geq 2 \dim|D|.$$

Equality $\Longleftrightarrow$ $D = 0$, $D = K$, or C is hyperelliptic and D is a multiple of the g^1_2 on C.

<u>Claim</u>: D is special. So Clifford's Thm applies.

$\Longrightarrow \deg(D) = 2 \dim|D|$.

If $K = D$, then we have

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & K \arrow[d] \arrow[dr, "q"] & & \\
0 \arrow[r] & \mathcal{O} \arrow[r] & E \arrow[r] \arrow[d] & K \arrow[r] & 0 \\
 & & \mathcal{O} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

If $q = 0$, then $K \to \mathcal{O}$, so q is an iso $\Rightarrow$ (1) splits.

Thus C is hyperelliptic.

<u>Pf of claim</u>: If $h^0(\mathcal{O}(K-D)) = 0$, then $H^0(\mathcal{O}(D)) \longrightarrow H^0(E)$ is an isomorphism

$$\Rightarrow H^0(\mathcal{O}(D)) \xrightarrow{\cong} H^0(E)$$

$$\begin{array}{ccc} \downarrow & & \twoheaddownarrow \\ \mathcal{O}(D) & \longrightarrow & E \end{array}$$

not surjective

Contradicts global generation of E.

Exercises:

1.) Prove Noether's Thm for $m > 2$.

2.) Show that if L is a l.b. of $\deg \geq 2g+1$ on a curve of genus g, then L is normally generated.

Example: (stability of rank 2 vector bundles on curves)

C a curve, L v.a.

$$\mathrm{Ext}^1(L, \omega_c) = \{0 \to \mathcal{O} \to E \to \underbrace{L \otimes \omega_C^*}_{= B} \to 0\}$$

parametrizes certain rank 2 v.b.s

$$\mathrm{Ext}^1(L, \omega_c) \cong H^1(\omega_c \otimes L^*) \cong H^0(L)^*$$

So $C \hookrightarrow \mathbb{P}(\mathrm{Ext}^1(L, \omega_c))$.

E unstable $\iff$ $E \twoheadrightarrow A$ line bundle s.t. $\deg A < \frac{1}{2}\deg B$

If E unstable, get diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O} & \to & E & \to & B & \to & 0 \\ & & & \searrow^{\neq 0} & \twoheaddownarrow & & & & \\ & & & & A & \leftarrow \deg = a < \frac{1}{2}\deg B & & & \end{array}$$

Section $\mathcal{O} \to A$ gives us a divisor $D = P_1 + \ldots + P_a$, and E corr. to a point on the span of the P_i's $\Rightarrow$ "Most" unstable bundles $(a=1)$ corr. to points on C. Next most unstable corr. to points on Σ, and so on.

<u>Exercise</u>: Explain correspondence.